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# Robust Control of Robot Manipulator with Actuators

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A Robust controller is designed for cascaded nonlinear uncertain systems that can be decomposed into two subsystems; that is, a series connection of two nonlinear subsystems, such as a robot manipulator with actuators. For such systems, a recursive design is used to include the second subsystem in the robust control. The recursive design procedure contains two steps. First, a fictitious robust controller for the first subsystem is designed as if the subsystem had an independent control. As the fictitious control, a nonlinear $H_\infty$ control using energy dissipation is designed in the sense of $L_2$-gain attenuation from the disturbance caused by system uncertainties to performance vector. Second, the actual robust control is designed recursively by Lyapunov's second method. The designed robust control is applied to a robotic system with actuators, in which the physical control inputs are not the joint torques, but electrical signals to the actuators.

**Key Words** : $H_\infty$ Control, Recursive Design, Nonlinear Matrix Inequality, Robot Manipulator

## 1. Introduction

For a class of nonlinear systems, in which every system is a series connection of a finite number of nonlinear subsystem, recursive design is applied to design stabilizing controls. Interesting progress in recursive design has been achieved in adaptive control of feedback linearizable systems (Kanellakoppoulos et al., 1991).

Since many systems inherently have uncertainties such as parameter variations, external disturbances, and unmodelled dynamics, robust control is considered in the recursive design. To design robust controllers, it is usual to use Lyapunov's second method, as proceeded in the existing results (Kim et al., 1999; Corless and Leitmann, 1981). One of the difficulties is that Lyapunov's second method requires a Lyapunov function for control design. Another robust control, which has attracted the attention of many researchers, is $H_\infty$ control. Although nonlinear $H_\infty$ control has been derived by an $L_2$-gain analysis based on the concept of energy dissipation (van der Schaft, 1992; Isidori and Astolifi, 1992), its applications are not easy due to the solution to the Hamilton-Jacobi Inequality (HJ Inequality). $H_\infty$ control problems in nonlinear systems reduce to the existence of the solution to HJ inequality and many methods have been proposed in recent papers (Astolifi and Lanari, 1994; Hu and Chang, 1988; Lu and Doyle, 1993; Park and Yim, 1999).

In the present paper, a robust control is designed for cascaded nonlinear uncertain systems using recursive design which is composed of two steps. In first step, a fictitious robust controller for the first subsystem is designed as if the subsystem had an independent control. As the fictitious control, nonlinear $H_\infty$ control is used in this work to guarantee the performance of the system to parameter uncertainty and the differ-

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entiability of the control, which is required to design the actual control recursively. The solution to the HJ inequality can be obtained through a more tractable nonlinear matrix inequality (NLMI) method and the fact that the matrices forming the NLMI are bounded (Park and Yim, 1999).

In the second step, the actual robust control is designed recursively by Lyapunov's second method. The designed control is applied to a robotic system with actuators. In a block diagram representation of the overall robotic system, the two blocks representing the actuator dynamics and the robot dynamics are connected in series.

This paper is organized as follows. In Sec. 2, the recursive design procedures are presented for certain and uncertain systems. In Sec. 3, a robust control is designed for a robotic system with actuators using the procedures of Sec. 2. In Sec. 4, simulations are performed to confirm the robust performance of the proposed controller for robot manipulator under parameter uncertainty. In Sec. 5, the conclusions are presented.

## 2. Recursive Design

### 2.1 Certain system

The class of nonlinear systems considered in this paper consists of those which are a series connection of two subsystems and whose dynamics are described by

$$\dot{x}_1 = f(x_1) + g(x_1)\varphi(x_2) \tag{1}$$

$$\dot{x}_2 = Ax_2 + Bu + h(x_1) \tag{2}$$

where $x_1$ and $x_2$ are the states of the systems and $u$ is a control input, and and $A$ are $B$ constant matrices. Equation (2) of the second subsystem is a differential equation whose output is the input signal to the first subsystem.

In recursive design, it is required that there exists a fictitious control which stabilizes the first subsystem, Eq. (1).

Assumption 1 (Global Stabilizability) There exists a $C^1$ control law $K(x_1)$ such that the equilibrium $x_1=0$ of the system $\dot{x}_1 = f(x_1) + g(x_1)K(x_1)$ is globally stable. This is established with a $C^1$ positive definite function $E(x_1)$ such that

$$\frac{\partial E}{\partial x_1}(f(x_1) + g(x_1)K(x_1)) \leq 0$$

In the first subsystem, this control law is not implementable and its effect must be achieved through the $x_2$-subsustem.

Assumption 2

1. There exists a constant matrix $K_w$ such that $\varphi(x_2) = K_w x_2$
2. There exists a fictitious state $x_w$ such that $K(x_1) = K_w x_w$.

Theorem 1 If Assumptions 1 and 2 are satisfied with the fictitious state $x_w$, then the overall system is feedback passive with respect to the new state

$$y = x_2 - x_w \tag{3}$$

and its passivity is achieved with the feedback control

$$u = B^{-1}(-Ax_2 - h(x_1) - \dot{x}_w - K_w^T(L_g E)^T + r) \tag{4}$$

Proof: Using Eq. (3) as a new state, we rewrite Eqs. (1) and (2) as

$$\dot{x}_1 = f(x_1) + g(x_1)K_w(x_w + y) \tag{5}$$

$$\dot{y} = Ax_2 + Bu + h(x_1) + \dot{x}_w \tag{6}$$

To show that the feedback control achieves passivity, we use a positive definite storage function such as

$$V(x_1, y) = E(x_1) + \frac{1}{2}y^T y.$$

With Eqs. (5) and (6), its time-derivative is

$$\dot{V} = \frac{\partial E}{\partial x_1}(f(x_1) + g(x_1)K_w x_w) + y^T(K_w^T(L_g E)^T + Ax_2 + Bu + h(x_1) + \dot{x}_w).$$

By Assumption 1 and 2, the feedback control law, Eq. (4), proves passivity.

With the additional feedback $r = -ky$, $k > 0$, its derivative is $\dot{V} \leq -ky^T y$. This proves global stability of its equilibrium $(x_1, y) = (0, 0)$.

### 2.2 Uncertain system

The recursive design in the previous section can be applied to a class of nonlinear systems that have uncertainties and require good tracking performance. Before proceeding with the detailed recursive design, the following Assumption 3 is needed to obtain the fictitious control when the nonlinear $H_\infty$ control is used as the fictitious control.

Assumption 3 There exists a $C^1$ control law Eq. (7) such that the system Eq. (8) is transformed to Eq. (9)

$$K(x_1, x_{1d}, u) = K'(x_1, x_{1d}) + u_f \quad (7)$$

$$\dot{x}_1 = f(x_1) + g(x_1) K(x_1, x_{1d}, u_f) \quad (8)$$

$$\dot{s} = F(x_1) s + G_1(x_1) w_1 + G_2(x_1) u_f \quad (9)$$

where $x_{1d} \in R^n$ is the desired trajectory of joints, $s(x_1, x_{1d}) \in R^n$ is the modified state, $w_1 \in R^w$ is the disturbance caused by uncertainties, and $u_f \in R^m$ is the control input for robustness.

### 2.2.1 Fictitious control

In the presence of disturbances, a robust control is needed as the fictitious control. A nonlinear $H_\infty$ control is designed as the fictitious control. Finding the nonlinear $H_\infty$ control is tantamount to finding a stabilizing state-feedback control input such that the closed-loop system has an $L_2$-gain equal to or less than $\gamma$ in the input-to-output sense. In nonlinear $H_\infty$ control design, it is essential to find the solution to the associated Hamilton-Jacobi (HJ) inequality derived from the input-output energy dissipation. If a solution exists, then it will guarantee stability as well as disturbance attenuation.

By Assumption 3 and with the performance vector the first subsystem, Eq. (1) can be described as

$$\dot{s} = F(x_1) s + G_1(x_1) w_1 + G_2(x_1) u_f \quad (10)$$
$$z = Hs + Du_f,\ H^T D = 0,\ D^T D > 0$$

where $F(x_1)$, $G_1(x_1)$, $G_2(x_1)$, $H$, $D$ are $C^0$ matrix -valued function of suitables dimensions.

In the form of Eq. (10), the derived HJ inequality is more tractable. The design of the nonlinear $H_\infty$ controller for the nonlinear system in the form of Eq. (10) is summarized as the following theorem (Park and Yim, 1999).

Theorem 2 Given $\gamma > 0$, suppose there exists a $C^0$ matrix-valued function $P$ satisfying

$$P^T F(x_1) + F^T(x_1) P + \frac{1}{\gamma^2} P^T G_1(x_1) G_1^T(x_1) P$$
$$+ H^T H - P^T G_2(x_1) [D^T D]^{-1} G_2^T(x_1) P \leq 0 \quad (11)$$

and there exists a non-negative function $E(s) \geq 0$ such that $\partial E / \partial s = 2 s^T P^T$. Then the control input satisfying $L_2$-gain $\leq \gamma$ is

$$u_f = -[D^T D]^{-1} G_2^T(x_1) Q^{-1} \quad (Q = P^{-1})$$

and the derivative of the storage function satisfies

$$\dot{E} \leq \gamma^2 \|w\|^2 - \|z\|^2$$

To obtain the solution to Eq. (11) easily, it is transformed to a nonlinear matrix inequality (NLMI) using the Schur complement :

$$\begin{bmatrix} FQ + Q^T F^T + (1/\gamma^2) G_1 G_1^T - G_2 [D^T D]^{-1} G_2 & Q^T \\ Q & -H^T H \end{bmatrix} \leq 0.$$

Solving the above NLMI yields a convex optimization problem. Unlike the linear case, this convex problem is not finite-dimensional. However, if the matrices forming the NLMI are bounded, then we only need to solve a finite number of LMIs.

The stabilizing control input at the first step becomes

$$K = K' - [D^T D]^{-1} G_2^T(x_1) Q^{-1} s$$

### 2.2.2 Real control

Assume there exists the fictitious state $x_{w1}$ and $x_{w2}$ satisfying Assumption 2; that is,

$$\begin{aligned} K &= K' + u_f \\ &= K_w (x_{w1} + x_{w2}) \\ &= K_w x_w \end{aligned}$$

It follows that the choice of $x_2 = x_w$, if permitted, would ensure robust stability. Since $x_2$ is not a control input, we cannot let $x_2 = x_w$. In the cascade, its effect must be achieved through the second subsystem.

Using $y = x_2 - x_w$ as a new state, we rewrite

$$\dot{s} = F(x_1) s + G_1(x_1) w + G_2(x_2) K_w (y + x_{w2})$$
$$\dot{y} = Ax_2 + Bu + h(x_1) - \dot{x}_w$$

To obtain the robust control, which achieves the $L_2$-gain property, we use the positive definite storage function

$$V(s, y) = E(s) + y^T y$$

Its time derivative is

$$\dot{V} \leq \gamma^2\|w\|^2 - \|z\|^2 + y^T(2K_w^T G_2^T Ps + Ax_2 + Bu + h(x_1) - \dot{x}_w) \leq \gamma^2\|w\|^2 - \|z\|^2 + y^T(\Delta A + Bu)$$

$\Delta A$ can be bounded as

$$\|\Delta A\| \leq \alpha_1\|s\| + \alpha_2\|x_2\| + \|h(x_1)\| + \|\dot{x}_{w1}\| + \|\dot{x}_{w2}\| \leq \rho(s, x_1, x_2, \dot{x}_{w1}, \dot{x}_{w2})$$

If it is assumed that $\underline{B} < \underline{B} < \bar{B}$ for known upper and lower bound matrices, the control input can be chosen as

$$u = -\underline{B}^{-1}\left[\frac{y}{\|y\|}\rho - y\right]$$

With the designed control input, the storage function satisfies $\dot{V} \leq \gamma^2\|w\|^2 - (\|z\|^2 + \beta\|y\|^2)$, $\beta > 0$, which achieves the $L_2$-gain property.

## 3. Actuator-Level Control of a Robot Manipulator

### 3.1 Dynamics of a robot manipulator with actuators

Consider the following dynamics of a robot manipulator actuated by permanent magnet DC motors:

$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + G(q) = K_\tau i \quad (12)$$

$$L_m\frac{di}{dt} + R_m i + K_m\dot{q} = v \quad (13)$$

where $q \in R^n$ is the joint position, $M(q) \in R^{n\times n}$ is the inertia matrix, and $N(q, \dot{q}) \in R^{n\times n}$ represents the centripetal and Coriolis torques. Symbols $R_m$ and $L_m$ denote the resistance and inductance of the armature circuit; $K_\tau$ and $K_m$ are the torque and back emf parameters of the motor; $i$ is the current in the armature of the motor; $v$ is the armature voltage. Eq. (13) of the actuator dynamics is a first-order differential equation whose output $i$ is the input signal to robot dynamics.

### 3.2 Fictitious control (Torque-Level Control)

#### 3.2.1 Transformation of dynamics

In an overall system representing the actuator dynamics and robot dynamics, the two subsystems are connected in series. That is, the overall system can be decomposed into two cascaded subsystems. The recursive design uses this structure.

Before proceeding with the detailed recursive design, a modified error for joint tracking which satisfy Assumption 3 is defined as

$$s = \dot{q} - \{\dot{q}_d - \Lambda(q - q_d)\} = \dot{q} - \dot{q}_r,$$

where $q_d$ and $\dot{q}_d$ are the desired position and velocity, respectively. If the elements of the vector approach zero as $t \to \infty$, so does the tracking error of the joints.

At the torque level of the robotic system, a suitable control input satisfying Assumption 3 can be chosen as

$$K = \hat{M}(q)\ddot{q}_r + \hat{N}(q, \dot{q})\dot{q}_r + \hat{G}(q) + u$$

Then Eq. (12) is transformed to

$$\dot{s} = F(q, \dot{q})s + G_1(q)w + G_2(q)u \quad (14)$$

where $F(q, \dot{q}) = -M^{-1}(q)N(q, \dot{q})$, $G_1(q) = M^{-1}(q)$, $G_2(q) = M^{-1}(q)$, and $w = \tilde{M}(q)\ddot{q}_r + \tilde{N}(q)\dot{q}_r + \tilde{G}$, is a disturbance vector caused by model uncertainties.

#### 3.2.2 The solution to the HJ inequality using LMI

To derive the HJ inequality for the robot manipulator dynamics transformed to affine form, each matrix term of Eq. (14) is substituted into Eq. (11). Then

$$-(MP^{-T})^{-1}N - N^T(P^{-1}M^T)^{-1} + H^TH + \frac{1}{\gamma^2}(MP^{-T})^{-1}(P^{-1}M^T)^{-1} - (MP^{-T})^{-1}(D^TD)^{-1}(P^{-1}M^T)^{-1} < 0$$

Premultiplying and postmultiplying the inequality by the positive definite matrices $MP^{-T}$ and $P^{-1}M^T$, respectively, the HJ inequality becomes

$$-NQM^T - MQ^TN^T + MQ^TH^THQM^T + \frac{1}{\gamma^2}I - (D^TD)^{-1} < 0 \quad (15)$$

where $Q = P^{-1}$. Using the Schur complement, Eq. (15) can be described as an NLMI as follows:

$$\begin{bmatrix} -NQM^T - MQ^TN^T + \frac{1}{\gamma^2}I - (D^TD)^{-1} & MQ^TH \\ HQM^T & -I \end{bmatrix} \leq 0 \tag{16}$$

The matrices $M(q)$ and $N(q, \dot{q})$ are nonlinear functions of $q$ and $\dot{q}$ in Eq. (16). However, those matrices include trigonometric functions and can be bounded when each joint velocity ranges between two empirically determined extreme values. Using this fact, we suppose that the matrices forming the above NLMI vary in some bounded sets of the space of matrices, i. e.,

$$[M(q), N(q, \dot{q}), H, D] \in Co\{[M_i, N_i, H, D]|_{i\in\{1,2,\cdots,L\}}\},$$

where $Co$ represents the convex hull.

Therefore, if

$$\begin{bmatrix} -N_iQM_i^T - M_iQ^TN_i^T + \frac{1}{\gamma^2}I - (D^TD)^{-1} & M_iQ^TH \\ HQM_i^T & -I \end{bmatrix} \leq 0$$

have a common solution $Q$ for all $i \in \{1, 2, \cdots, L\}$ then $Q$ is also a solution to Eq. (16), and the stabilizing robust control input is determined as

$$u_f = -(D^TD)^{-1}G_2^TQ^{-1}s$$

### 3.3 Real control input

The total control input at the torque level becomes

$$\begin{aligned} K &= \hat{M}(q)\ddot{q}_r + \hat{N}(q, \dot{q})\dot{q}_r + \hat{G}(q) \\ &\quad - (D^TD)^{-1}G_2^TQ^{-1}s \\ &= K_\tau(i_{w1} + i_{w2}) \\ &= K_\tau i_w \end{aligned}$$

Using $y = i - i_w$ as a new coordinate, we rewrite

$$\begin{aligned} \dot{s} &= -M^{-1}(q)N(q, \dot{q})s + M^{-1}(q)w \\ &\quad + M^{-1}(q)K_\tau i_{w2} + M^{-1}(q)K_\tau y \\ \dot{y} &= -L_m^{-1}R_m i - L_m^{-1}K_m\dot{q} + L_m^{-1}v - \frac{di_w}{dt} \end{aligned}$$

To design the robust control, which achieves the $L_2$-gain property, we use the positive definite storage function

$$V(s, y) = E(s) + y^Ty$$

Its time derivative is

$$\dot{V} \leq \gamma^2\|w\|^2 - \|z\|^2 + y^T(\Delta A + L_m^{-1}v)$$

where $\Delta A = 2K_\tau M^{-T}Ps - L_m^{-1}R_m i - L_m^{-1}K_m\dot{q} - \frac{di_w}{dt}$.

$$\begin{aligned} \|\Delta A\| &\leq \alpha_1\|s\| + \alpha_2\|i\| + \alpha_3\|\dot{q}\| + \left\|\frac{di_w}{dt}\right\|: \\ &= \rho(s, i, \dot{q}, \frac{di_w}{dt}) \end{aligned}$$

If it is assumed that $\underline{L} \leq L \leq \bar{L}$ for known upper and lower bound matrices, the control input can be chosen as

$$v = -\bar{L}\left[\frac{y}{\|y\|}\rho - y\right]$$

With the designed control input, the storage function satisfies $\dot{V} \leq \gamma^2\|w\|^2 - (\|z\|^2 + \beta\|y\|^2)$, $\beta > 0$ which achieves $L_2$-gain property. To smooth out the control discontinuity, the saturation-type control can be chosen as

$$v = -\bar{L}\left[\frac{y}{\|y\| + \varepsilon\exp(-\delta t)}\rho - y\right]$$

where $\varepsilon$ and $\delta$ are positive constants.

## 4. Simulation

Robust control using a recursive method is designed for a two-degree-of-freedom planar robot manipulator with actuators. The Simulation was performed under parameter uncertainties. The objective of the simulation is to show the enhancement of robustness to parameter uncertainty. The set of dynamic parameters is summarized in Tables 1 and 2. As an extreme disturbance, the mass of link 2 is assumed to vary by 50% at 2 seconds. The system model matrices forming LMIs are determined by the bounds of parameter uncertainty and the trigonometric functions. The LMIs for the matrix $Q$ are solved using an efficient convex algorithm in the Matlab toolbox. It should be noted that ease in controller

**Table 1** Manipulator parameters used in the simulation

| | Real Length | Real Mass | Bound of Mass |
|---|---|---|---|
| Link1 | 0.5 m | 2 kg | [1.5, 2.5] |
| Link2 | 0.3 m | 1 kg | [0.5, 1.5] |

**Table 2** Actuator parameters used in the simulation

| | Real value | Bound |
|---|---|---|
| Inductance | 0.05 H | [0.03, 0.07] |
| Resistance | 0.3 $\Omega$ | [0.2, 0.5] |
| Back emf Constant | 0.065 V sec/rad | [0.055, 0.075] |
| Torque Constant | 2Nm/A | [1.9, 2.1] |

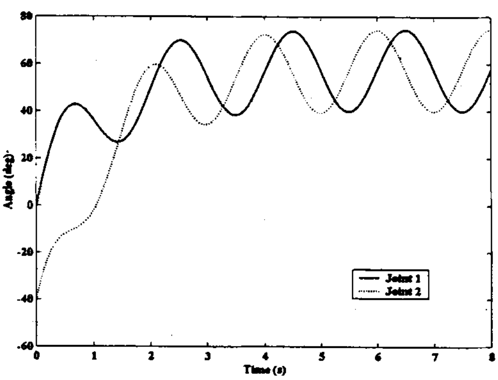

**Fig. 1** Desired trajectories of joints

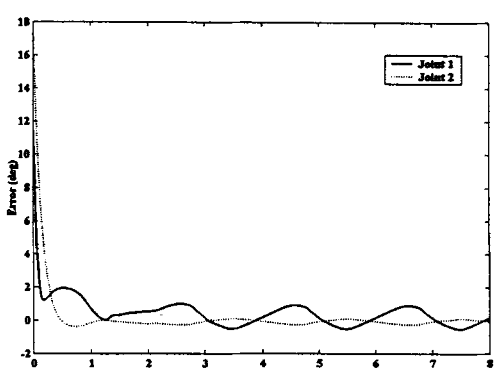

**Fig. 2** Position errors of joints

tuning can be obtained since the solution of the LMIs, if any, is found easily by an optimization algorithm.

The joints of the manipulator are commanded to trace trajectories shown in Fig. 1 with some initial errors. The initial errors of the joints are 11.45° and 17.19°, respectively. The estimates of the manipulator model matrices in Eq. (12) are assumed to be $\bar{M}=0$ and $\bar{N}=0$. The estimate of the gravity torque $G$ is determined from the equation in the dynamics using the estimates of

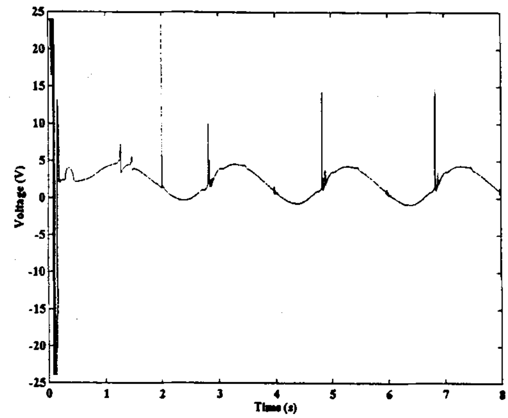

**Fig. 3** Voltage of actuator of joint 1

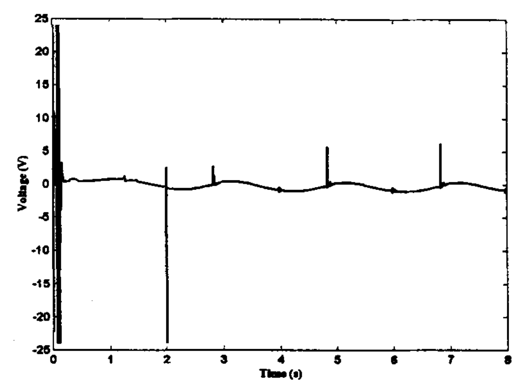

**Fig. 4** Voltage of actuator of joint 2

mass $\bar{m}_1=1.8kg$ and $\bar{m}_2=0.8kg$.

The position error and input voltage are shown in Fig. 2~4, respectively. Though the control input is saturated at the initial state, the proposed controller shows satisfactory robustness performance even under large parameter uncertainties.

## 5. Conclusion

Using recursive design, a robust control is designed for nonlinear uncertain systems, which can be decomposed into two cascaded subsystems. First, a fictitious robust controller is designed using nonlinear $H_\infty$ control. The associated HJ inequality is transformed to an NLMI and its approximated solution is obtained from the fact that the terms in the matrices can be bounded. The application of the proposed method is simple since the gain matrix can be obtained easily by an efficient convex optimization algorithm. Second, the actual robust control is designed recursively through the design of a fictitious control. The designed robust control is applied to a robotic system with actuators.

## References


Astolfi, A. and Lanari, L., 1994, "$H_\infty$ Control of Rigid Robots," *IFAC Robot Control*, pp. 171~176.

Boyd, S., Ghaoui, E. E., Feron, E. and Balakrishnan, V., 1994, *Linear Matrix Inequalities in System and Control Theory*, SIAM books.

Corless, M. J. and Leitmann, G., 1981, "Continuous State Feedback Guaranteeing Uniform Boundedness for Uncertain Dynamic Systems," *IEEE Trans. Automat, Contr.*, Vol. 26, pp. 1139~1144.

Hu, S. S. and Chang, B. C., 1988, "Design of a Nonlinear $H_\infty$ Controller for the Inverted Pendulum System," *Proc. of IEEE International Conference on Control Applications*, pp. 699~703.

Isidori, A. and Astolfi, A., 1992, "Disturbance Attenuation and $H_\infty$-Control via Measurement Feedback in Nonlinear Systems," *IEEE Trans. Automat. Contr.*, Vol. 37, pp. 1283~1293.

Kanellakoppoulos, I., Kokotovic, P. V. and Morse, A. S., 1991, "Systematic Design of Adaptive Controllers for Feedback Linearizable Systems," *IEEE Trans. Automat. Contr.*, Vol. 36, pp. 1241~1253.

Kim, D. H., Kang, J. Y. and Lee, K. I., 1999, "Nonlinear Robust Control Design for a 6 DOF Parallel Robot," *KSME International Journal*, Vol. 13, pp. 557~568.

Lewis, F. L., Abdallab, C. T. and Dawson, D. M., 1993, *Control of Robot Manipulator, Macmillan.*

Lu, W. M. and Doyle, J. C., 1993, "$H_\infty$ Control of Nonlinear Systems: a Convex Charateriza tion," Caltech CDS Tech. Memo., No. CTI-CDS-93-020.

Park, J. H. and Yim, J., 1999, "Nonlinear $H_\infty$ Control of Robotic Manipulators," *Proc. of IEEE Conference on Systems, Man and Cybernetics*, pp. 866~871.

Qu, Z. and Dawson, D. M., 1996, *Robust Tracking Control of Robot Manipulators*, IEEE Press.

Sepulchre, R., Jankovic, M. and Kokotovic, P., 1997, *Constructive Nonlinear Control*, Springer.

van der Schaft, A. J., 1992, "L2-Gain Analysis of Nonlinear Systems and Nonlinear State Feedback," *IEEE Trans. Automat. Contr.*, Vol. 37, pp. 770~784.